

Gauge and Topological Symmetries in the Bulk Quantization of Gauge Theories

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A gauge theory with 4 physical dimensions can be consistently expressed as a renormalizable topological quantum field theory in 5 dimensions. We extend the symmetries in the 5-dimensional framework to include not only a topological BRST operator s that encodes the invisibility of the “bulk” (the fifth dimension), but also a gauge BRST operator w that encodes gauge-invariance and selects observables. These symmetries provide a rich structure of Ward identities which assure the renormalizability of the theory, including non-renormalization theorems. The 5-dimensional approach considerably simplifies conceptual questions such as for instance the Gribov phenomenon and fermion doubling. A confinement scenario in the 5-dimensional framework is sketched. We detail the five-dimensional mechanism of anomalies, and we exhibit a natural lattice discretization that is free of fermion doubling.

1. Introduction

In [1] we gave arguments for the relevance of a five-dimensional representation of gauge theories in four-physical dimensions. We proved the existence of a local quantum field theory that is perturbatively renormalizable by power counting and free of the Gribov ambiguity. We also gave its lattice formulation, and suggested that the 5-dimensional framework naturally avoids fermion doubling. A possible interpretation is that the fifth-dimension is the stochastic time that Parisi and Wu proposed a long time ago for stochastically quantizing the Yang–Mills theory [2], and the 4-dimensional physical theory lives in any given chosen time-slice of the space with five dimensions. However the beauty of the resulting theory suggests that the fifth time plays a more fundamental and more general role for example in elucidating the Gribov and fermion doubling problems. In particular we consider the 5-dimensional functional integral to be more fundamental than the Langevin or Fokker-Planck equations.

After the discovery of the four-dimensional Yang–Mills topological quantum field theories [3], it was realized that the supersymmetric formulation of stochastic quantization also determines a topological field theory [4]. This was an early example of a physical theory that lives in the “boundary” of a space with an additional dimension, independently of the process in the “bulk” that determines the many possible ways the a Fokker–Planck distribution converges to an equilibrium distribution. One recognizes a holographic phenomenon at work here, and various connections between different theories have been exhibited in this way, with the idea that stochastic quantization is analogous to a Stokes theorem for the path integral [5]. With the inclusion of fermions, and BRST ghosts, the local 5-dimensional formulation transcends its purely stochastic origin. From now on, we call it bulk quantization.

One must establish the consistency of quantization with an additional time, and the four-dimensional Yang–Mills theory is an important and challenging case. Long before the invention of topological field theory, it was shown that the Parisi–Wu conjecture [2] is compatible at the perturbative level with the Faddeev–Popov method [6]. For reviews of stochastic quantization, see [7], [8] and [9]. Further development of stochastic quantization, particularly gauge-invariant stochastic regularization, may be found in [4], [10] and [11], and for renormalization in [12]. For renormalization of non-gauge theories in the 5-dimensional formulation, see [13].

The hope of enriching our perspectives is of course at the non-perturbative level. The 5-dimensional theory possesses a supersymmetry of the topological type which ensures

that the expectation-value, once equilibrium is achieved, is independent of the details of the initial conditions. The existence of an unobservable fifth time considerably simplifies many conceptual problems that perplex the sole and too narrow four-dimensional perspective. For example the local quantum field theory in 5 dimensions avoids the question of Gribov copies that jeopardizes the Faddeev–Popov prescription in four dimensions [14]. As shall be discussed elsewhere, the introduction of the additional time allows one to replace problematic gauge fixing by an appropriate *gauge transformation* in such a way that the Gribov question becomes irrelevant. The idea of looking at gauge theories from five dimensions can be put in correspondence with the description of conformal theories from the Chern–Simons action in three dimensions.

Our definition of observables in [1] was not entirely satisfactory to the extent that it was not based on an invariance principle. Here we will fill this hole in our presentation, and introduce, in addition to the supersymmetry operator s that expresses quantization with a fifth time [13], another BRST-symmetry operator, w , that implements gauge symmetry in 5 dimensions and which is compatible with s . This necessitates introducing an additional field, in a way that is inspired by an idea originally due to Horne [15] in the context of the topological Yang–Mills theory in four dimensions, and emphasized in subsequent works using the idea of equivariant cohomology [16]. The 5-dimensional action of the Yang–Mills theory is entirely defined by the requirement of w and s symmetry. We will define observables by the cohomology of w taken at a fixed but arbitrary time-slice.

There are two features of the 5-dimensional formulation of 4-dimensional quantum field theory – whether of ϕ^4 or of Yang-Mills type – which we wish to emphasize. The first is that even though the action is s -exact, $I = sX$, where s has all possible characteristics of a topological BRST operator, we are *not* in the context of a topological theory of the usual type, as was noted in [1]. The reason is that the observables O are *not* required to be s -invariant, $sO \neq 0$. Indeed the cohomology of s is empty, so if the observables were s -invariant, they would be s -closed, $O = sY$, and would have vanishing expectation-value, because our theory has no zero modes that would allow $\langle sY \rangle \neq 0$. Rather, in the case of a scalar theory, the observables are all possible correlators taken at equal time, and in the case of gauge theories, they are also required to be in the cohomology of another BRST operator w , such that $w^2 = sw + ws = 0$. Beyond technical details and subtleties, w is the expression of the gauge symmetry in the five-dimensional framework. The five-dimensional theory is thus not a topological theory, although its s -exact action looks topological. Rather, our interpretation is that the latter property is the simplest

way to assure that possible renormalization constants are the same as in the 4-dimensional theory. The second feature, which holds both for gauge and scalar theories, is that the correlators are distributions in 5 dimensions, and in general they have singularities at equal times. As a result, physical observables, which are restricted to a time slice, may not be well-defined. Indeed, as shown in Appendix E, the correlator of three chiral currents, $\langle j(x_1, t_1)j(x_2, t_2)j(x_3, t_3) \rangle$, is ambiguous in the equal-time limit, and this is the origin of the triangle anomaly in the 5-dimensional formulation. We expect that this kind of obstruction to consistent equal-time limits for the observables occurs only when the gauge theory is anomalous.

We will explain in a separate paper that the condition of fixed time relies on the correspondence of Schwinger–Dyson equations in 4 and 5 dimensions.

The 5-dimensional formulation accommodates in a local description the gauges that are actually used at present in lattice gauge theory for numerical evaluation. These gauges, such as the minimal Landau, minimal Coulomb or maximal Abelian gauges, are fixed by minimizing an appropriately chosen functional. They cannot be correctly described by the Faddeev–Popov method that is characterized by a local gauge condition, such as $\partial_\lambda A_\lambda = 0$, that does not distinguish between minima and saddle points of the minimizing functional. Nevertheless these gauges are represented by a local action in 5 dimensions. We would like to emphasize that the 5-dimensional local theory does not reproduce the gauge-non-invariant part of the standard 4-dimensional Faddeev–Popov formulation. The latter can only be reached from the 5-dimensional theory by means of a non-local action [6], and then only at the perturbative level.

Instead of a local gauge-fixing in 4 dimensions which is known not to exist [17], the 4-dimensional probability distribution is obtained from the solution of a Fokker–Planck equation in 5 dimensions that preserves the 4-dimensional probability at each instant. The solution of the Fokker–Planck equation, which determines all gauge-invariant correlation functions in 4 dimensions is represented by a local 5-dimensional gauge theory of topological type. Its path integral formula is valid non-perturbatively, and we have previously given its BRST-invariant lattice regularization [1].

The organization of the paper is as follows. The basic formulation of the theory is presented in sec. 2. Here the field content is explained, the s - and w -symmetries are defined, and the most general renormalizable s - and w -invariant action is exhibited. It is important in this regard to keep in mind that the fifth time has engineering dimension double of that of ordinary space-time coordinates $[\partial/\partial t] = 2[\partial/\partial x_\mu] = 2$. In sec. 3, we

briefly sketch a physical interpretation of the theory. A number of further developments of the theory and various issues are explored in the Appendices. In Appendix A, we derive the Ward identities which assure the stability of the theory under renormalization and fix a number of renormalization constants. In particular it is shown that gA_5 is invariant under renormalization in the minimal Landau gauge, and thus may be used to define an invariant charge in QCD. In Appendix B, properties of the minimal Landau-gauge are derived. This gauge provides the frame for a confinement scenario that is described in Appendix C, in which long-range “forces” are transmitted by A_5 . In Appendix D, the theory is extended to Dirac spinor fields. The important topic of anomalies is addressed in Appendix E. Here a 5-form is found which is a candidate for an obstruction in the 5-dimensional theory. We show however that the familiar triangle anomaly of the 4-dimensional theory has another origin: it is a singularity that appears when fifth times are set equal, as is necessary to obtain the physical 4-dimensional correlators. In Appendix F, we show that fermion doubling may be avoided by lattice discretization of the 5-dimensional theory.

2. Field content and symmetries

We first focus on the pure Yang–Mills case, and will introduce coupling to spinors later. The 5-dimensional action used in [1]

$$I = \int dx^\mu dx^5 \, s \, \text{Tr}(\bar{\Psi}^\mu(F_{5\mu} - D_\lambda F_{\lambda\mu} - \frac{1}{2}b_\mu) + \dots) \quad (2.1)$$

is s -exact, and \dots will be specified shortly. (2.1) looks like a topological action. Indeed, s is a topological BRST operator that will be defined shortly, as well as all relevant fields. Greek indices denote 4-dimensional Euclidean components, $\lambda, \mu = 1, \dots, 4$. The first term in (2.1) is invariant under 5-dimensional gauge transformations. Roughly speaking, it concentrates the path integral around the solutions of the equation $F_{5\mu} - D_\lambda F_{\lambda\mu} = 0$. The five-dimensional gauge symmetry of the action will be broken in a BRST-invariant way by means of

$$aA_5 = \partial_\lambda A_\lambda, \quad (2.2)$$

where a is a gauge parameter. This condition should not be interpreted as a Faddeev–Popov gauge-fixing. As will be discussed elsewhere, $A_5 = \dot{g}g^{-1}$ is the generator of a time-dependent gauge transformation $g(x, t)$ that acts on A_μ for $\mu = 1, \dots, 4$, and as such it can be fixed arbitrarily, apart from the constraints imposed by renormalizability. Indeed

the role of A_5 as the generator of a time-dependent gauge transformation survives the algebraic quantization of the 5-dimensional theory, as is shown in Appendix A. The successful avoidance of the Gribov problem is reflected in the fact that the ghost propagators are parabolic when (2.2) is enforced in a BRST-invariant way, so the ghost propagators are retarded, $G(x, t) = 0$ for $t < 0$, and consequently closed ghost loops vanish and the Faddeev–Popov determinant is trivial. Heuristically, one recognizes that (2.2) is like an axial gauge in A_5 , for which there is no Gribov ambiguity, and the infinite range of the variable $t = x_5$ avoids the Singer theorem [17].

When the ghost and auxiliary fields are integrated out, one obtains the action of stochastically quantized gauge theory [2],

$$I = \int dx^\mu dx^5 \text{Tr}(F_{5\mu}^2 + (D_\lambda F_{\lambda\mu})^2) \quad (2.3)$$

with stochastic gauge-fixing [14].

The observables of the theory were defined in [1] in the following intuitive way: they are correlation functions of gauge-invariant functions of the gauge field components, A_μ , for $\mu = 1, \dots, 4$, taken at equal values of $t = x_5$. The Green functions can be first computed and renormalized at different values of time, and then one takes the limit of equal times. From the point of view of quantum field theory the fifth time appears as a regulator and from a geometrical point of view of topology, it appears as a variable that enlarges the space and simplifies topological properties. We shall refine the definition of observables here, and define observables from a symmetry principle, in order to have a better control of their renormalization properties. This will lead us to refine our knowledge of the symmetry of (2.1). We want to make this notion precise, and end up with the definition of all observables in the cohomology of a certain symmetry operator, w , which is compatible with s .

We have shown in [1] that the action (2.1) gives a perturbatively renormalizable theory. Although the action (2.1) is not $SO(5)$ invariant, its BRST symmetry is $SO(5)$ invariant, which is sufficient for a consistent description, since the five-dimensional description is holographic, and the only things that matters is to recovers the $SO(4)$ invariance for the observables. Power-counting constrains the way the invariance under $SO(5)$ symmetry is reduced down to $SO(4)$: indeed, the canonical dimension in mass units of each one of the four-dimensional components of the gauge field is unity whereas A_5 has dimension two. (The Yang–Mills coupling constant has dimension zero).

Let us now define all the fields, which depend on x^μ and $t = x_5$ and are Lie algebra valued. New fields will occur, as compared to [1] because of the w -symmetry. The bose

and fermi fields are in one-to-one correspondence, and are related by the topological BRST operator s , as in a supersymmetric theory. (This suggests the possibility of a link between the topological BRST operator s and some kind of Poincaré twisted supersymmetry.) In addition to A_μ and A_5 there are: c , which has the quantum numbers the ordinary Faddeev–Popov ghost but plays a different role; Ψ_μ and Ψ_5 , the topological Fermi vector ghosts corresponding to A_μ and A_5 ; and Φ , a commuting ghost of ghost. These fields have respectively ghost number $N_s = 1, 1, 1$ and 2 . We also have corresponding anti-ghosts \bar{c} , $\bar{\Psi}_\mu$ and $\bar{\Phi}$, with ghost number $N_s = -1, -1$ and -2 respectively, and Lagrange multipliers b_μ , l and $\bar{\eta}$, with ghost number $N_s = 0, 0$ and -1 respectively. (In [1] we used $\bar{c} = \bar{\Psi}_5$ and $l = b_5$.) See the diagram below for the relations of these fields and their quantum numbers.

There are five degrees of freedom for the choice of a dynamics, one for each component of A . The Lagrange multiplier field l serves to impose the condition (2.2), and the b_μ ’s enforce the Langevin equation. Finally $\bar{\eta}$ is a fermion Lagrange multiplier for the gauge fixing of the longitudinal modes in Ψ .

The topological BRST operator s is not relevant for defining observables. Indeed, the cohomology of s with ghost number zero is empty. Although it involves the gauge symmetry with a ghost of ghost phenomenon to ensure its nilpotency, it only represents the irrelevance of the details of the process in the bulk.

To encode gauge-invariance and distinguish gauge-invariant observables, we need another BRST operator, which we call w . We thus introduce a second ghost number N_w and new fields. The total grading is the sum $N_s + N_w + p$, where p is the ordinary form degree. We need a second Faddeev–Popov ghost λ and, as will be explained shortly, also its ghost of ghost μ . The ghost-field λ plays the role of the ordinary Faddeev–Popov ghost, and in the “effective” theory on the boundary, the w operator induces the ordinary BRST symmetry. One actually has a quartet of additional ghosts and anti-ghosts λ , μ , $\bar{\lambda}$, $\bar{\mu}$, which maintains supersymmetry. All the fields and ghosts besides λ , μ , $\bar{\lambda}$ and $\bar{\mu}$ have ghost number $N_w = 0$. The latter have respectively ghost number $N_w = 1, 1, -1, -1$, and $N_s = 0, 1, 0, -1$. It is convenient to use a bigrading notation $\phi^{(N_s, N_w)}$ that indicates the ghost numbers of any given field ϕ . With this notation one has $(s\phi)^{(N_s+1, N_w)}$ and $(w\phi)^{(N_s, N_w+1)}$, $(ws\phi)^{(N_s+1, N_w+1)}$. All ghost numbers can be read from the following diagram:

$$\begin{array}{ccccc}
& & A_\mu^{(00)}, A_5^{(00)} & & \\
& \swarrow & & \searrow & \\
\Psi_\mu^{(1,0)}, \Psi_5^{(1,0)} & & & & \bar{\Psi}_\mu^{(-1,0)} \\
c^{(1,0)}, \lambda^{(0,1)} & & & & \bar{c}^{(-1,0)} \\
& & & & \bar{\lambda}^{(0,-1)} \\
\swarrow & & \searrow & \swarrow & \\
\Phi^{(2,0)}, \mu^{(1,1)} & & b_\mu^{(0,0)} & & \bar{\Phi}^{(-2,0)}, \bar{\mu}^{(-1,-1)} \\
& & l^{(0,0)} & & \\
& & & \searrow & \\
& & & & \bar{\eta}^{(-1,0)}
\end{array}
\tag{2.4}$$

The engineering dimensions (in mass units) of the fields are assigned according to:

$$\begin{aligned}
[c] &= [\lambda] = [\mu] = [\Phi] = 0 \\
\left[\frac{\partial}{\partial x^\mu}\right] &= [A_\mu] = [\Psi_\mu] = 1 \\
\left[\frac{\partial}{\partial x^5}\right] &= [A_5] = [\Psi_5] = 2 \\
[b_\mu] &= [\bar{\Psi}_\mu] = 3 \\
[l] &= [\bar{c}] = [\bar{\mu}] = [\bar{\lambda}] = [\bar{\Phi}] = [\bar{\eta}] = 4.
\end{aligned}
\tag{2.5}$$

These values will play a key role in the determination of the five-dimensional action. The asymmetry in the dimensions of the fields with ordinary labels μ and 5 is absolutely crucial, and will explain how the good properties of the power counting in the conventional four-dimensional formulation of gauge theories are still present in the five-dimensional formulation. Moreover, since the anti-ghosts have dimension 3 or 4 and $\partial/\partial t$ has dimension 2, all ghost actions will be parabolic and all ghost propagators will be retarded.

In (2.4), the arrows conveniently relate the fields that can be transformed into each other by s - or w -transformations that we will give shortly, in accordance with the separate conservation of the ghost numbers N_s and N_w . The diagram (2.4) exhibits the boson-fermi pairing which always occurs in theories with a “topological” symmetry. The subtlety is the way the symmetry is expressed, with a delicate separation of the gauge symmetry transformations from the topological transformations, expressed respectively by the w - and s -transformation. By convention, since the 2 ghost numbers are conserved, we assign dimension in such a way that both s and w leave dimension unchanged.

We now define the action of s and w .¹ We require

$$w^2 = s^2 = sw + ws = 0. \quad (2.6)$$

This property is automatically assured by defining the symmetry in the following geometrical way, which is by now a standard in the BRST paradigm:

$$\begin{aligned} (s + w + d)(A + c + \lambda) + \frac{1}{2}[A + c + \lambda, A + c + \lambda] &= F + \Psi_\mu dx^\mu + \Psi_5 dx^5 + \Phi \\ (s + w + d)(F + \Psi_\mu dx^\mu + \Psi_5 dx^5 + \Phi) &= -[A + c + \lambda, F + \Psi_\mu dx^\mu + \Psi_5 dx^5 + \Phi] \end{aligned} \quad (2.7)$$

The separate conservation of the ghost numbers N_s and N_w determines the action of s and w on all fields except $s\lambda$ and wc , since the geometrical equation only determines

$$s\lambda + wc = -[\lambda, c]. \quad (2.8)$$

To resolve this degeneracy we introduce the ghost μ , with $N_s = N_w = 1$, and impose $s\lambda = \mu$. The result of the decomposition is:

$$\begin{aligned} sA_\mu &= \Psi_\mu + D_\mu c, & s\Psi_\mu &= D_\mu \Phi - [c, \Psi_\mu], & sc &= \Phi - \frac{1}{2}[c, c], \\ sA_5 &= \Psi_5 + D_5 c, & s\Psi_5 &= D_5 \Phi - [c, \Psi_5], & s\Phi &= -[c, \Phi], \\ s\lambda &= \mu, & s\mu &= 0 \end{aligned} \quad (2.9)$$

$$\begin{aligned} wA_\mu &= D_\mu \lambda, & w\Psi_\mu &= -[\lambda, \Psi_\mu], & wc &= -\mu - [\lambda, c], \\ wA_5 &= D_5 \lambda, & w\Psi_5 &= -[\lambda, \Psi_5], & w\Phi &= -[\lambda, \Phi] \\ w\lambda &= -\frac{1}{2}[\lambda, \lambda] & w\mu &= -[\lambda, \mu] \end{aligned} \quad (2.10)$$

Observe that w acts on A_μ and A_5 like an infinitesimal gauge transformation with gauge parameter λ , and observables will be required to be w -invariant. Because of the inhomogeneous term μ in $wc = -\mu - [\lambda, c]$, w -invariance of a quantity with ghost-number zero assures that it is independent of c . Moreover μ is a topological ghost for c and consequently w -invariant observables are independent of c .

For the anti-ghosts of the s -symmetry, $\bar{\Psi}_\mu$, $\bar{\Phi}$, and the corresponding Lagrange multipliers b_μ and $\bar{\eta}$, one has:

$$\begin{aligned} (s + w)\bar{\Psi}_\mu + [c + \lambda, \bar{\Psi}_\mu] &= b_\mu \\ (s + w)b_\mu + [c + \lambda, b_\mu] &= [\Phi, \bar{\Psi}_\mu] \end{aligned} \quad (2.11)$$

¹ A more symmetric diagram would involve the idea of anti- s and anti- w symmetries

$$\begin{aligned}
(s+w)\bar{\Phi} + [c + \lambda, \bar{\Phi}] &= \bar{\eta} \\
(s+w)\bar{\eta} + [c + \lambda, \bar{\eta}] &= [\Phi, \bar{\Phi}]
\end{aligned}
\tag{2.12}$$

(Notice that $(s+w)^2 = 0$ amounts to $(s+w + [c = \lambda, \cdot])^2 = [\Phi, \cdot]$) This definition implies:

$$s\bar{\Psi}_\mu = -[c, \bar{\Psi}_\mu] + b_\mu, \quad sb_\mu = -[c, b_\mu] + [\Phi, \bar{\Psi}_\mu] \tag{2.13}$$

$$w\bar{\Psi}_\mu = -[\lambda, \bar{\Psi}_\mu], \quad wb_\mu = -[\lambda, b_\mu] \tag{2.14}$$

$$s\bar{\Phi} = -[c, \bar{\Phi}] + \bar{\eta}, \quad s\bar{\eta} = -[c, \bar{\eta}] + [\Phi, \bar{\Phi}] \tag{2.15}$$

$$w\bar{\Phi} = -[\lambda, \bar{\Phi}], \quad w\bar{\eta} = -[\lambda, \bar{\eta}] \tag{2.16}$$

Finally we give the action of w and s on the anti-ghosts and Lagrange multipliers \bar{c} , l , $\bar{\mu}$ and $\bar{\lambda}$. We take:

$$\begin{aligned}
(s+w)\bar{\mu} &= \bar{c} + \bar{\lambda} \\
(s+w)(\bar{c} + \bar{\lambda}) &= 0
\end{aligned}
\tag{2.17}$$

In order to break the degeneracy of the geometrical equation $w\bar{\lambda} + s\bar{c} = 0$, we introduce the new Lagrange multiplier field l , and impose $s\bar{c} = -l$. After expansion in the ghost numbers N_s and N_w we obtain,

$$\begin{aligned}
s\bar{c} &= -l, \quad sl = 0 \\
s\bar{\mu} &= \bar{\lambda}, \quad s\bar{\lambda} = 0
\end{aligned}
\tag{2.18}$$

$$\begin{aligned}
w\bar{\mu} &= \bar{c}, \quad w\bar{c} = 0 \\
w\bar{\lambda} &= l, \quad wl = 0
\end{aligned}
\tag{2.19}$$

We are now ready to ask the standard question, within the BRST paradigm, of determining the gauge-fixed action, the possible counter-terms, and the anomalies of a theory that is gauge and Lorentz invariant and renormalizable by power counting. Here we have the two symmetries w and s , with $(w+s)^2 = 0$, and we must classify the local functions of the fields that are both w - and s -invariant. The important question of anomalies is addressed in Appendix E, where we display an intriguing new cocycle stemming from (2.19). In what follows we consider the question of determining the action in 5 dimensions from the requirement of s - and w -symmetry.

We wish to find the most general solution to the equations $sI = 0$ and $wI = 0$, where $I = \int dt d^4x \mathcal{L}$, and \mathcal{L} is a local Lagrangian density of engineering dimension 6, and $I = I^{(0,0)}$. Here we use the notation defined above to indicate the N_s and N_w ghost

quantum numbers. We rely on the fact that the cohomology of s with ghost number zero is empty. Thus the action $I^{(0,0)}$ must be an s -exact term, $I^{(0,0)} = sI^{(-1,0)}$. We shall use the strategy of descent equations, with d and s replaced respectively by s and w , because our s , like d , has empty cohomology. By w -invariance of $I^{(0,0)}$ we have $0 = wI^{(0,0)} = wsI^{(-1,0)} = -swI^{(-1,0)}$. Since the cohomology of s is empty, this gives

$$wI^{(-1,0)} = sI^{(-2,1)}. \quad (2.20)$$

Upon multiplying this equation by w , we obtain $0 = wsI^{(-2,1)} = -swI^{(-2,1)}$, and so, since the cohomology of s is empty, we have

$$wI^{(-2,1)} = sI^{(-3,2)}. \quad (2.21)$$

Now we use the fact that we have assigned the engineering dimensions to the fields in such a way that the operators s and w preserve engineering dimension, so the $I^{(i,j)}$ that we have introduced all have engineering dimension $[I^{(i,j)}] = 0$, and all the corresponding densities have engineering dimension 6. Since $I^{(-3,2)}$ has ghost number $N_s = -3$, it must contain either at least 3 factors of the anti-ghost fields with $N_s = -1$, or else at least one power of the anti-ghost field $\bar{\Phi}^{(-2,0)}$ and another anti-ghost field. In the first case the corresponding density has engineering dimensions 9 or greater because all anti-ghost fields have dimension 3 or 4, as one sees from (2.5). In the second case the corresponding density has engineering dimension 7 or greater. We conclude that $I^{(-3,2)} = 0$, so from (2.21), we have $wI^{(-2,1)} = 0$. Thus $I^{(-2,1)}$ is of the form

$$I^{(-2,1)} = I_{\text{inv}}^{(-2,1)} + wI^{(-2,0)}, \quad (2.22)$$

where $I_{\text{inv}}^{(-2,1)}$ is an element of the cohomology of w . However there is no local density in the cohomology of w , of engineering dimension 6, with these ghost quantum numbers. For $I_{\text{inv}}^{(-2,1)}$ must contain at least one power of either $\mu^{(1,1)}$ or $\lambda^{(0,1)}$. If it contains $\mu^{(1,1)}$, then it must contain so many anti-ghost fields that its engineering dimension is too high. On the other hand if it contains $\lambda^{(0,1)}$, then there is no such element of the cohomology of w . (For example $(D_5 \bar{\Phi})^a \lambda^a$ has the right dimension, but is not in the cohomology of w .) We conclude that $I_{\text{inv}}^{(-2,1)} = 0$, so from (2.22) we obtain $I^{(-2,1)} = wI^{(-2,0)}$. We substitute this into (2.20) and obtain $wI^{(-1,0)} = swI^{(-2,0)}$, or

$$w(I^{(-1,0)} + sI^{(-2,0)}) = 0. \quad (2.23)$$

Thus the quantity in parenthesis is of the form

$$I^{(-1,0)} + sI^{(-2,0)} = I_{\text{inv}}^{(-1,0)} + wI^{(-1,-1)}, \quad (2.24)$$

where $I_{\text{inv}}^{(-1,0)}$ is an element of the cohomology of w . Upon substitution of this equation into $I^{(0,0)} = sI^{(-1,0)}$, we obtain

$$I^{(0,0)} = sI_{\text{inv}}^{(-1,0)} + swI^{(-1,-1)}. \quad (2.25)$$

It follows that the Lagrangian density must be of the form

$$s \text{Tr}[\bar{\Psi}^\mu K_\mu^{(0,0)}(A, b_\mu) + \bar{\Phi} K^{(1,0)}(\Psi, A)] + ws \text{Tr}[\bar{\mu} L^{(0,0)}(A)], \quad (2.26)$$

where $K_\mu^{(0,0)}(A, b_\mu)$, $K^{(1,0)}(\Psi, A)$ and $L^{(0,0)}(A)$ have dimension 3, 2 and 2 respectively. Here we have used the fact that the fields c , $\bar{\lambda}$ and $\bar{\mu}$ undergo transformations of topological type under the infinitesimal gauge transformation w , so they cannot appear in the cohomological term $I_{\text{inv}}^{(-1,0)}$. In fact $I_{\text{inv}}^{(-1,0)}$ must be constructed out of combinations of local fields that transform covariantly under w . Note that the only possible w -exact term that is s -invariant and thus s -exact is sw -exact. To obtain this result we used ghost number conservation and power counting arguments for a local Lagrangian density of engineering dimension 6.

One gets by inspection that the two first term in (2.26), up to multiplicative renormalization constants, must be:

$$I_{\text{inv}} = \int dx^\mu dx^5 s \text{Tr}[\bar{\Psi}^\mu (F_{5\mu} - D_\lambda F_{\lambda\mu} - \frac{1}{2}b_\mu) + \bar{\Phi}(a'\Psi_5 - D_\lambda \Psi_\lambda)]. \quad (2.27)$$

This determines the gauge-invariant part of the action, or more precisely the part of the action that is in the non-trivial part of the cohomology of w with ghost number zero and dimension 6. The first term, corresponds to the 5-dimensional action of stochastic quantization that one can guess from the Langevin equation for the four-dimensional classical Yang–Mills action. It may be written as:

$$I_1 = \int dx^\mu dx^5 s \text{Tr}[\bar{\Psi}^\mu (F_{5\mu} + \frac{\delta S}{\delta A_\mu} - \frac{1}{2}b_\mu)], \quad (2.28)$$

where $S = \int d^4x (1/4)F_{\mu\nu}^2$ is the standard Yang–Mills action. The second term of (2.27), that is linear in $\bar{\Phi}$, fixes in a gauge-covariant way the internal gauge invariance of Ψ_μ and Ψ_5 . These terms, that we derived in a straightforward way from the requirement

of locality and compatibility with power counting, show the conceptual limitation of the idea of stochastic quantization in its original formulation. We refer to [1] for more details concerning the meaning of each term that occurs in the expansion of (2.27).

For a local action of dimension 6, the only possibility for the last term in (2.26) is

$$I_{\text{gf}} = \int dx^\mu dx^5 \, w s \, \text{Tr}([\bar{\mu}(aA_5 - \partial \cdot A)] , \quad (2.29)$$

and the total action is given by

$$I = I_{\text{inv}} + I_{\text{gf}} . \quad (2.30)$$

Remarkably, the only possible gauge-fixing term with dimension 6 provides a linear gauge-fixing. (If the coupling to matter were included, the generalization of the Feynman-'tHooft gauge for spontaneously broken symmetries is easily obtained, $aA_5 = \partial_\mu A_\mu + v\phi$.) The derivation of (2.29) using w -invariance is a significant improvement compared to the derivation in [1]. The term $I_{\text{inv}} = sI_{\text{inv}}^{(-1,0)}$ is the same as in [1], and its expansion may be found there.

To see in detail how (2.29) solves the question of raising the degeneracy with respect to ordinary gauge transformations of (2.27), we expand

$$\begin{aligned} w s [\bar{\mu}(aA_5 - \partial_\nu A_\nu)] = & w \{ \bar{\lambda}(aA_5 - \partial_\nu A_\nu) \\ & + \bar{\mu}[a\Psi_5 + aD_5 c - \partial_\nu(\Psi_\nu + D_\nu c)] \} . \end{aligned} \quad (2.31)$$

The first term gives

$$l(aA_5 - \partial_\nu A_\nu) - \bar{\lambda}(aD_5 \lambda - \partial_\nu D_\nu \lambda), \quad (2.32)$$

which fixes the gauge for A_5 and λ . The second term in (2.31) gives:

$$\begin{aligned} & \bar{c}[a\Psi_5 + aD_5 c - \partial_\nu(\Psi_\nu + D_\nu c)] \\ & + \bar{\mu}\{aD_5 \mu - \partial_\nu D_\nu \mu\} + a[\lambda, \Psi_5 + D_5 c] - \partial_\nu[\lambda, (\Psi_\nu + D_\nu c)] \} . \end{aligned} \quad (2.33)$$

The equation of motion of \bar{c} determines a certain linear combination of c and Ψ_5 . But a similar situation occurs for $\bar{\eta}$. Using its equation of motion from I_{inv} , Ψ_5 is determined, so both c and Ψ_5 are determined. This can be seen by doing the translation $\bar{\eta} \rightarrow \bar{\eta} - \bar{c}$. Finally, the equation of motion of $\bar{\mu}$ determines μ , because its equation of motion is parabolic.

An essential feature is that the equations of motion of all ghosts is parabolic, of the type $(\partial_t - a^{-1}\partial_\nu^2 + \dots)\xi = 0$. It follows that the free ghost propagators $G_0(x, t) = \theta(t)(\frac{a}{4\pi t})^2 \exp(\frac{-ax^2}{4t})$ are all retarded, and consequently *all closed ghost loops vanish*. The

only possible exception is the tadpole which gives a purely local contribution to the action which is a renormalization counter-term. However even the tadpole contribution vanishes with dimensional regularization. With dimensional regularization in mind, we conclude that the determinant of each ghost is unity.

Observables O are required to be in the cohomology of w , thus $wO = 0$ and $O \neq w(X)$, with ghost number 0, and lying in a fixed but arbitrary time slice. However, as noted in the Introduction, they are not s -invariant, $sO \neq 0$. We may impose additional conditions on the class of observables. The most conservative policy would be to allow only functions of the variables that are present in the 4-dimensional theory, namely the A_μ for $\mu = 1, \dots, 4$, and Dirac spinor fields. Expectation-values of observables are independent of the parameters of the w -exact term, that is on the parameter a . Their independence of the other parameter a' is quite clear, as shown in [1]: closed ghost loops vanish, and so cannot contribute to the expectation values of observables, and the latter cannot depend on a' . Since observables are w -invariant, which implies that they are gauge invariant, we do get the observables of the 4-dimensional theory.

As for a detailed perturbative proof of the stability of renormalization, it can be done quite rigorously, using the Ward identities derived in Appendix A. If one uses dimensional regularization by computing correlators in $5 - \epsilon$ dimensions, one finds that the action I undergoes a multiplication renormalization for all its fields and parameters, as a result of imposing the Ward identities of both w - and s -invariance in the gauge determined by (2.32), that is with the gauge condition (2.2). It should be understood that the correspondence between equal-time correlators in 5 dimensions and those in 4 dimensions is for the physical observables only (defined just above, from the w -cohomology).²

3. Interpretation of the action

One way to exhibit the physical content of the theory is to integrate out all ghost and auxiliary fields. We will do this in two steps, in order to get an intermediate result which will be useful later. The first step is to integrate out all ghost fields except λ and $\bar{\lambda}$. As explained above, closed ghost loops vanish because ghost propagators are retarded,

² The comparison for gauge non-invariant gauge functions would require a non-local theory in five dimensions, [6], and presents no interest since locality is a key tool for mastering perturbative renormalizability.

$D(x, t) = 0$ for $t < 0$, so the ghost determinant which results from this integration is a constant, and the action (2.30) reduces to

$$I = \int d^5x [ib_\mu(F_{5\mu} - D_\lambda F_{\lambda\mu}) + (1/2)b_\mu^2 + il(aA_5 - \partial_\mu A_\mu) + \bar{\lambda}(aD_5 - \partial_\mu D_\mu)\lambda] . \quad (3.1)$$

This action possesses the ordinary BRST invariance, here implemented by w , that encodes gauge invariance in five dimensions. The s -invariance that stabilizes the action (2.30) has already been exploited, and (3.1) is supposed to be already renormalized. Notice also that the elimination of Ψ and $\bar{\Psi}$ deprived us of the Ito term that is useful non-perturbatively [1].

We next translate $b_\mu \rightarrow b_\mu - iD_\lambda F_{\lambda\mu}$. The cross-term $F_{5\mu}D_\lambda F_{\lambda\mu}$ is an exact derivative because of the Bianchi identity, and we obtain

$$I = \int d^5x [ib_\mu F_{5\mu} + (1/2)b_\mu^2 + (1/2)(D_\lambda F_{\lambda\mu})^2 + il(aA_5 - \partial_\mu A_\mu) + \bar{\lambda}(aD_5 - \partial_\mu D_\mu)\lambda] . \quad (3.2)$$

If one identifies b_μ as a 4-component color-“electric” field, this resembles Faddeev–Popov theory in first-order formalism, with a particular gauge-fixing term, except that the magnetic energy $(1/2)B_i^2$ is replaced by $(1/2)(D_\lambda F_{\lambda\mu})^2$. This action still has w -invariance, but s -invariance is lost.

As a second step we integrate out the remaining ghost and auxiliary fields $\lambda, \bar{\lambda}, b_\mu$, and b_5 . Noting that the ghost determinant is again a constant, we obtain the 5-dimensional non-Abelian gauge action in second-order formalism

$$I[A] = \int d^5x [(1/2)(\partial_5 A_\mu - a^{-1}D_\mu \partial_\lambda A_\lambda)^2 + (1/2)(D_\lambda F_{\lambda\mu})^2] . \quad (3.3)$$

Here A_5 is gauge fixed to $A_5 = a^{-1}\partial_\lambda A_\lambda$, a gauge condition often referred to as *stochastic* gauge-fixing [14]. Indeed the functional integral associated with the above 5-dimensional action represents the solution of a Fokker-Planck or Langevin equation. The latter describe a stochastic process that may be simulated numerically, and in which the variable $t = x_5$ counts sweeps over a 4-dimensional lattice.

The striking feature of the present approach is that the ghost determinant is unity as a result of the parabolic ghost equation. This is a strong indication that the Gribov ambiguity is not a difficulty in the present formulation. However we have not directly addressed here the issue of establishing that the action (3.3) is a valid quantization of the 4-dimensional Yang–Mills theory. We shall return to this topic on another occasion.

4. Conclusion

We have reconsidered the quantization of 4-dimensional gauge theories in 5 dimensions with particular attention to the invariance needed to characterize physical observables. We implemented this invariance by a BRST-operator w , with $w^2 = 0$, that encodes ordinary gauge invariance, and physical operators O are required to satisfy $wO = 0$, and $O \neq w(X)$. The operator w is to be distinguished from the topological BRST-operator s , with $s^2 = 0$, that will occur in the quantization of all theories with an additional time, including gauge and non-gauge theories. The two operators are compatible in the sense that $(s + w)^2 = 0$. Together, these two symmetries are extremely restrictive, and we have constructed the most general action I , eqs. (2.27) and (2.29), that is invariant under both symmetries $sI = wI = 0$, and that is renormalizability by power counting. In Appendix A we derived the Ward identities associated with these 2 symmetries. In the other Appendices we have examined various aspects and extensions of the theory such as the Landau-gauge limit, a confinement scenario, spinor fields, anomalies and lattice discretization without fermion doubling.

The esthetics of the formulation in five dimensions, and its attractive geometric interpretation, strongly suggest that it should be adopted as a starting point for defining a gauge theory. As we have seen in our discussion of anomalies, Appendix E, the fifth time acts as a regulator. Indeed the theory produces correlation functions in five dimensions, and the physical limit requires taking a slice in the fifth time. However new divergences appear when the times coincide (as is obvious in momentum space where setting the times equal corresponds to additional integrations over the conjugate momenta). This is our interpretation of the chiral anomaly.

The 5-dimensional formulation provides the stage for a simple confinement scenario in the minimal Landau gauge. In this scenario, the long-range confining “force” is transmitted by A_5 , the fifth component of the gluon field. It is suggestive that the imposition of the analog of Gauss’s law in the fifth dimension can produce a confinement scenario similar to the one in the Coulomb gauge [18] and [19], where the long-range confining force is transmitted by A_4 . However the scenario is compatible with manifest 4-dimensional Lorentz invariance.

We use the s and w invariance to improve the argument that we gave in [1] to prove renormalizability of the action in 5 dimensions. This requires that we also address possible anomalies. We exhibit an interesting cocycle in 5 dimensions (solution of the Wess–Zumino

consistency conditions for s and w symmetries). However its coefficient appears to be 0 in general, because of the topological nature of the action. Consequently the Noether currents K_M ($M = 1, \dots, 5$) of the 5-dimensional action are strictly conserved. The origin of the 4-dimensional anomaly may be understood when one recognizes that the currents of physical interest are not the K_M , but rather the Noether currents J_μ ($\mu = 1, \dots, 4$) of the 4-dimensional action. These two different currents are not related in any obvious way. Indeed the J_μ are not conserved in the 5-dimensional theory in general. However consistency requires that the J_μ generate the appropriate 4-dimensional Ward identities when inserted into correlation functions at a fixed fifth time. This may fail due to singularities that appear in the correlation functions when the times are set equal. Indeed we have verified by explicit computation that the ABBJ triangle anomaly appears as a discontinuity of the 3-point J -current correlator as equal times are approached. Power counting in 5-dimensions leads us to expect that there are no other such breakdowns.

The principles of locality, gauge symmetry and power counting also determine the form of the topological action for spinors, whether chiral or not. It is gratifying that this yields a convergent functional integral that gives well-defined correlation functions. It should be emphasized that the stochastic interpretation of the fifth coordinate must be abandoned when the local five-dimensional theory is extended to include fermions. Remarkably, the 5-dimensional spinor action allows a natural lattice discretization that does not suffer from fermion doubling, as is shown in Appendix F. It is not necessary to introduce a domain wall for this purpose [20], [21], [22].

In summary, the five-dimensional formulation of renormalizable gauge theories appears as a very powerful tool for investigating the non perturbative questions relative to the Yang–Mills theory.

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Appendix A. Ward Identities, Renormalization, and Stability of the Action

We shall show that with the action (3.1) the combination $g\lambda$ is invariant under renormalization,

$$g\lambda = g_r\lambda_r \tag{A.1}$$

Moreover in the Landau gauge limit $a \rightarrow 0$, the quantity gA_5 is invariant also invariant under renormalization.

In [19] and [23] it was shown that gA_4 is invariant under renormalization in the Coulomb gauge in the 4-dimensional formulation. For the same reason gA_5 is invariant under renormalization in 5-dimensional formulation in the Landau gauge. [The result depends on the following argument. The component j_5 of the conserved Noether current of the w -symmetry is given by $j_5 = \partial\mathcal{L}/\partial(\partial_5 A_\mu) D_\mu \lambda = ib_\mu D_\mu \lambda$, because all derivatives with respect to $t = x_5$ are contained in $F_{5,\mu} = \partial_5 A_\mu - D_\mu A_5$. Moreover the left-hand side of Gauss's law is given by $\delta I/\delta A_5 = iD_\mu b_\mu$. This allows us to write the conserved Noether charge $Q = \int d^4x j_5$ as $Q = - \int d^4x \lambda \delta I/\delta A_5$. No product of fields appears in this expression, so it provides a Ward identity that is special to this gauge, and it leads to the condition $Z_{A_5} Z_g = 1$ on the renormalization constants Z_{A_5} and Z_g of A_5 and g .]

The renormalization of the model contains some interesting features that we discuss in the present appendix. First, we will show that the rich content of the symmetry (the s -symmetry and the w -symmetry (2.10)) provides strong constraints on the form of the gauge fixed action; second, we will show that the particular gauge choice (2.29) implies non-renormalization properties for the ghost fields.

In order to discuss the set of functional equations which implement the symmetries at the quantum level, we introduce the sources coupled to the variations of the fields. The latter are needed to renormalize theories with non-linear transformations as in our case. Given ϕ , a field of the set $\{A_\mu, A_5, \Psi_\mu, \Psi_5, \Phi, c, \bar{\Psi}_\mu, b_\mu, \bar{\Phi}, \bar{\eta}\}$, we introduce its (ws)-source ϕ^* , its w -source ϕ' , and its s -source ϕ'' . The statistic of ϕ', ϕ'' is opposite to that of ϕ , and the statistic of ϕ^* is the same of ϕ . On the other hand, the ghost numbers of ϕ', ϕ'', ϕ^* are easily fixed by their couplings. In particular for a field ϕ with ghost numbers (N_s, N_w) , we have

$$\begin{aligned} I_{\text{sources}}[\phi] &= \int d^5x \sum_{\phi} w \left[s(\phi^* \phi) \right] = \int d^5x \sum_{\phi} w \left[(s \phi^*) \phi + (-)^{N_s + N_w} \phi^* s \phi \right] = \\ &= \sum_{\phi} \int d^5x \left[(ws \phi^*) \phi + (-)^{N_s + N_w + 1} (s \phi^*) w \phi + (-)^{N_s + N_w} (w \phi^*) s \phi + \phi^* ws \phi \right]. \end{aligned} \quad (\text{A.2})$$

Adding a ws -trivial term does not modify either the s - or the w -cohomology of the classical theory. However, at the quantum level, we have to translate the s - and w -transformations in terms of nilpotent functional operators, and therefore, we also require

$$\begin{aligned} s\phi^* &= (-)^{1+N_s+N_w} \phi', & w\phi^* &= (-)^{N_s+N_w} \phi'', & ws\phi^* &= -sw\phi^* = 0, \\ s\phi' &= 0, & w\phi' &= 0, & s\phi'' &= 0, & w\phi'' &= 0. \end{aligned} \quad (\text{A.3})$$

Consequently, the eq. (A.2) becomes

$$I_{\text{sources}}[\phi] = \sum_{\phi} \int d^5x \left[\phi' (w\phi) + \phi'' (s\phi) + \phi^* (ws\phi) \right]. \quad (\text{A.4})$$

As concerns the fields λ, μ and the anti-ghosts $\bar{\lambda}, \bar{\mu}, \bar{c}$, due to the fact that their s and w -transformations are almost trivial, we can spare several sources, but we cannot avoid the following two terms

$$I_{\text{ghost}} = \int d^5x \left[\lambda' \left(-\frac{1}{2} [\lambda, \lambda] \right) - \lambda^* [\lambda, \mu] \right], \quad (\text{A.5})$$

where we identify λ^* with μ' . More specifically, we do not introduce sources for the antighost fields $\bar{\lambda}, \bar{c}$ and $\bar{\mu}$, and for the fields λ and μ , we have $s\lambda^* = -\lambda'$ and $w\lambda^* = 0$ which is consistent with the identification $\lambda^* = \mu'$. In the following, the symbol I denotes the action including the source terms.

Following the conventional procedure [24] and given the source terms (A.4) and (A.5), we can establish the functional identities for s - and w -symmetry for the generating functional Γ of irreducible Green functions

$$\begin{aligned} \mathcal{S}(\Gamma) &= \int d^5x \left\{ \sum_{\phi} \left[\frac{\delta\Gamma}{\delta\phi} \frac{\delta\Gamma}{\delta\phi''} - (-)^{N_s+N_w} \phi' \frac{\delta\Gamma}{\delta\phi^*} \right] + \mu \frac{\delta\Gamma}{\delta\lambda} - \lambda' \frac{\delta\Gamma}{\delta\lambda^*} + \bar{\lambda} \frac{\delta\Gamma}{\delta\bar{\mu}} - l \frac{\delta\Gamma}{\delta\bar{c}} \right\} = 0, \\ \mathcal{W}(\Gamma) &= \int d^5x \left\{ \sum_{\phi} \left[\frac{\delta\Gamma}{\delta\phi} \frac{\delta\Gamma}{\delta\phi'} - (-)^{N_s+N_w} \phi'' \frac{\delta\Gamma}{\delta\phi^*} \right] + \frac{\delta\Gamma}{\delta\lambda} \frac{\delta\Gamma}{\delta\lambda'} + \frac{\delta\Gamma}{\delta\mu} \frac{\delta\Gamma}{\delta\mu'} + \bar{c} \frac{\delta\Gamma}{\delta\bar{\mu}} + l \frac{\delta\Gamma}{\delta\bar{\lambda}} \right\} = 0, \end{aligned} \quad (\text{A.6})$$

where the sum is extended to the entire set of fields ϕ of the present model. The nilpotency and the commutation relations between the s -symmetry and the w -symmetry are expressed by the following equations

$$\mathcal{S}_{\Gamma}^2 = 0, \quad \mathcal{W}_{\Gamma}^2 = 0, \quad \left\{ \mathcal{W}_{\Gamma}, \mathcal{S}_{\Gamma} \right\} = 0, \quad (\text{A.7})$$

where \mathcal{S}_{Γ} and \mathcal{W}_{Γ} are the linearized version of the functional operators involved in (A.6) and are defined by $\mathcal{S}_{\Gamma}(\Xi) \equiv \frac{\partial}{\partial\epsilon} \mathcal{S}(\Gamma + \epsilon\Xi)|_0$. Besides eqs. (A.6), one has to take into account the equations of motion for the Lagrangian multiplier l (cf. eq. (2.29))

$$\frac{\delta\Gamma}{\delta l} = aA_5 - \partial_{\mu}A^{\mu}, \quad (\text{A.8})$$

where a is a gauge parameter. From the five-dimensional point of view, the gauge fixing (A.8) appears like a Landau gauge fixing type, but from the four-dimensional point of

view, A_5 plays the rôle of the Lagrangian multiplier and the gauge fixing (A.8) is a truly Lorentz gauge type. In the following we will show that, for a generic a the theory satisfies a set of functional equations for the ghost fields λ, c, μ . Those equations imply the non-renormalization of the ghost fields themselves and, in the case $a = 0$, a further equation implies the non-renormalization of the combination gA_5 field.

The existence of a solution to the system of equations (A.8), (A.6) is based on the Fröbenius theorem whose main hypothesis is the existence of an algebra of vector fields on the manifold spanned by functionals like Γ . The algebra is constructed by computing the commutation relations between the eqs (A.6) (cf. (A.7)) and with the eq. (A.8). For a generic functional \mathcal{F} , they read

$$\begin{aligned}\mathcal{W}_{\mathcal{F}}\left(\frac{\delta\mathcal{F}}{\delta l} - aA_5 + \partial_\mu A^\mu\right) - \frac{\delta}{\delta l}\mathcal{W}(\mathcal{F}) &= \frac{\delta\mathcal{F}}{\delta\bar{\lambda}} - a\frac{\delta\mathcal{F}}{\delta A'_5} + \partial_\mu\frac{\delta\mathcal{F}}{\delta A'_\mu}, \\ \mathcal{S}_{\mathcal{F}}\left(\frac{\delta\mathcal{F}}{\delta l} - aA_5 + \partial_\mu A^\mu\right) - \frac{\delta}{\delta l}\mathcal{S}(\mathcal{F}) &= \frac{\delta\mathcal{F}}{\delta\bar{c}} - a\frac{\delta\mathcal{F}}{\delta A''_5} + \partial_\mu\frac{\delta\mathcal{F}}{\delta A''_\mu},\end{aligned}\tag{A.9}$$

and, by commuting again the resulting equations with the operators $\mathcal{S}_{\mathcal{F}}, \mathcal{W}_{\mathcal{F}}$, we have

$$\begin{aligned}\mathcal{W}_{\mathcal{F}}\left(\frac{\delta\mathcal{F}}{\delta\bar{c}} - a\frac{\delta\mathcal{F}}{\delta A''_5} + \partial_\mu\frac{\delta\mathcal{F}}{\delta A''_\mu}\right) - \left(\frac{\delta}{\delta\bar{c}} - a\frac{\delta}{\delta A''_5} + \partial_\mu\frac{\delta}{\delta A''_\mu}\right)\mathcal{W}(\mathcal{F}) &= \frac{\delta\mathcal{F}}{\delta\bar{\mu}} - a\frac{\delta\mathcal{F}}{\delta A^*_5} + \partial_\mu\frac{\delta\mathcal{F}}{\delta A^*_\mu}, \\ \mathcal{S}_{\mathcal{F}}\left(\frac{\delta\mathcal{F}}{\delta\bar{\lambda}} - a\frac{\delta\mathcal{F}}{\delta A'_5} + \partial_\mu\frac{\delta\mathcal{F}}{\delta A'_\mu}\right) - \left(\frac{\delta}{\delta\bar{\lambda}} - a\frac{\delta}{\delta A'_5} + \partial_\mu\frac{\delta}{\delta A'_\mu}\right)\mathcal{S}(\mathcal{F}) &= \frac{\delta\mathcal{F}}{\delta\bar{\mu}} - a\frac{\delta\mathcal{F}}{\delta A^*_5} + \partial_\mu\frac{\delta\mathcal{F}}{\delta A^*_\mu}.\end{aligned}\tag{A.10}$$

All the other commutation relations are trivial and therefore (A.6), (A.9), (A.10) describe all the possible non-vanishing commutation relations and the algebra is, in fact, closed. We notice that the commutation relations between the ghost equations (A.9) with the operators $\mathcal{S}_{\mathcal{F}}, \mathcal{W}_{\mathcal{F}}$ generate the same functional equation for the ghost μ . This fact is a consequence of the anti-commutation relations between $\mathcal{S}_{\mathcal{F}}$ and $\mathcal{W}_{\mathcal{F}}$. Finally, the equations (A.9), (A.10) expressed in term of Γ can be easily integrated. This amounts to a redefinition of the sources A'_M, A''_M , and A^*_M . We will not discuss these details since they are common to the conventional procedure of the BRST quantization with linear gauge fixings.

More interesting and specific to the present model are further functional equations for the antighost fields $\bar{\lambda}, \bar{c}$ and $\bar{\mu}$. These can be derived by analyzing the corresponding tree level equations and observing that the composite operators which appear in those formulae

are already present in the extended action I . From eqs. (2.31), (2.32), (2.33), and (A.4), we have

$$\begin{aligned}
\frac{\delta I}{\delta \lambda(x)} &= \left(aD_5 \bar{\lambda} - D_\mu \partial^\mu \bar{\lambda} \right) - a \left[\Psi_5 + D_5 c, \bar{\mu} \right] - \left[\Psi_\mu + D_\mu c, \partial^\mu \bar{\mu} \right] + [\lambda', \lambda] + [\lambda^*, \mu] \\
&\quad + \sum_\phi \int d^5 y \left[(-)^{1+N_s+N_w} \phi' \frac{\delta}{\delta \lambda(x)} (w\phi)(y) + (-)^{N_s+N_w} \phi^* \frac{\delta}{\delta \lambda(x)} (ws\phi)(y) \right], \\
\frac{\delta I}{\delta c(x)} &= \left(aD_5 \bar{c} - D_\mu \partial^\mu \bar{c} \right) + aD_5 [\bar{\mu}, \lambda] - D_\mu [\partial^\mu \bar{\mu}, \lambda] + [c', \lambda] \\
&\quad + \sum_\phi \int d^5 y \left[(-)^{N_s+N_w+1} \phi'' \frac{\delta}{\delta c(x)} (s\phi)(y) + (-)^{N_s+N_w} \phi^* \frac{\delta}{\delta c(x)} (ws\phi)(y) \right. \\
&\quad \left. + (-)^{N_s+N_w+1} \phi' \frac{\delta}{\delta c(x)} (w\phi)(y) \right], \\
\frac{\delta I}{\delta \mu(x)} &= \left(aD_5 \bar{\mu} - D_\mu \partial^\mu \bar{\mu} \right) - [\lambda^*, \lambda] - c' + \sum_\phi \int d^5 y \left[\phi^* \frac{\delta}{\delta \mu(x)} (ws\phi)(y) \right].
\end{aligned} \tag{A.11}$$

It is important to note that the terms proportional to the sources ϕ', ϕ'' are linear in the quantum fields and therefore they do not require an independent renormalization besides the usual field renormalization. As an example, some of those terms are explicitly shown in the last equation. On the other hand, the last terms proportional to the sources ϕ^* are not linear in quantum fields and therefore they require more care. Eqs. (A.11) are not suitable for quantization since they involve the renormalization of operators like $(aD_5 \bar{\lambda} - D_\mu \partial^\mu \bar{\lambda})$ which, unfortunately, do not belong to the set of those coupled to the sources (A.4) and (A.5). However, considering the integrated (over the 5-dimensional manifold) version of eqs. (A.11) and integrating by parts, we have the following equations for Γ :

$$\begin{aligned}
\int d^5 x \frac{\delta \Gamma}{\delta \lambda(x)} &= \int d^5 x \left\{ \left[\bar{\lambda}, \frac{\delta \Gamma}{\delta l} \right] + \left[\bar{\mu}, \frac{\delta \Gamma}{\delta \bar{c}} \right] + [\mu', \mu] + [\lambda', \lambda] \right. \\
&\quad \left. + \sum_\phi [\phi', \phi] + \left[\phi^*, \frac{\delta \Gamma}{\delta \phi''} \right] \right\}, \\
\int d^5 x \frac{\delta \Gamma}{\delta c(x)} &= \int d^5 x \left\{ \left[\bar{c}, \frac{\delta \Gamma}{\delta l} \right] + \left[\bar{\mu}, \frac{\delta \Gamma}{\delta \bar{\lambda}} \right] + \left[\lambda, \frac{\delta \Gamma}{\delta \mu} \right] + [c', \lambda] \right. \\
&\quad \left. + \sum_\phi [\phi'', \phi] + \left[\phi^*, \frac{\delta \Gamma}{\delta \phi'} \right] \right\}, \\
\int d^5 x \frac{\delta \Gamma}{\delta \mu(x)} &= \int d^5 x \left\{ \left[\bar{\mu}, \frac{\delta \Gamma}{\delta l} \right] - [\lambda^*, \lambda] - c' + \sum_\phi (-)^{N_s+N_w} [\phi, \phi^*] \right\}.
\end{aligned} \tag{A.12}$$

Notice that, by using the sources ϕ' and ϕ'' , we are able to translate from the tree level approximation (A.11) to the quantum level also the terms coming from the sw variations. As a check of the procedure, we can compute the anti-commutation relations between the anti-ghost equations (A.12) and the ghost equations (A.9) and (A.10), to see that they close on the gauge-fixing equation (A.8) and they are compatible with each other. This supports the hypothesis of the Fröbenius theorem which allows us to integrate the complete system of equations. In particular, it is important to compute the commutation relation between the \mathcal{W}_Γ and the first equation of (A.12) or that of \mathcal{S}_Γ with the first one. This gives a new functional equation

$$\begin{aligned} \int d^5x \Big\{ & \left[\bar{\lambda}, \frac{\delta\Gamma}{\delta\bar{\lambda}} \right] + \left[\bar{\mu}, \frac{\delta\Gamma}{\delta\bar{\mu}} \right] + \left[\bar{c}, \frac{\delta\Gamma}{\delta\bar{c}} \right] + \left[\lambda, \frac{\delta\Gamma}{\delta\lambda} \right] + \left[\mu, \frac{\delta\Gamma}{\delta\mu} \right] \\ & + \sum_{\phi} \left[\phi, \frac{\delta\Gamma}{\delta\phi} \right] + \left[\phi', \frac{\delta\Gamma}{\delta\phi'} \right] + \left[\phi'', \frac{\delta\Gamma}{\delta\phi''} \right] + \left[\phi^*, \frac{\delta\Gamma}{\delta\phi^*} \right] + [c', \mu] \Big\} = 0, \end{aligned} \quad (\text{A.13})$$

which implements the invariance under the rigid gauge transformations of the model. As in the case of Yang–Mills, quantized with the conventional BRST technique, in the Landau gauge, rigid gauge invariance is a by-product of the dynamics of the ghost fields [25].

Equations (A.12) control the renormalization of the ghost fields λ, c, μ and, in particular, by means of those equations the ghost fields have no independent renormalization. This is a common feature of Landau type gauge fixing [25].

To derive these non-renormalization properties, and for pedagogical purposes, we consider a model without chiral fermions and we assume an invariant regularization scheme. In that framework, we can renormalize the model multiplicatively. That is, all the fields will be renormalized by a suitable wave function renormalization $\phi \rightarrow Z_\phi \phi$. Requiring that eqs. (A.8), (A.9), (A.10), and (A.12) are preserved by the renormalization procedure, we immediately get

$$\begin{aligned} Z_a &= Z_A Z_5^{-1}, & Z_{\bar{\mu}} &= Z_{\bar{c}} = Z_{\bar{\lambda}} = Z_l = Z_A^{-1}, \\ Z_\mu &= Z_l Z_{\bar{\mu}}^{-1} = 1, & Z_\lambda &= Z_l Z_{\bar{\lambda}}^{-1} = Z_{\bar{\mu}}^{-1} Z_{\bar{c}} = 1, \\ Z_c &= Z_l Z_{\bar{c}}^{-1} = Z_{\bar{\mu}}^{-1} Z_{\bar{\lambda}} = Z_\mu Z_\lambda^{-1} = 1, \end{aligned} \quad (\text{A.14})$$

where we used the rescaled gauge fields $A_M \rightarrow g A_M$. Due to the relations $Z_\lambda = Z_c = Z_\mu = 1$, it is direct to conclude that the products $g\lambda, gc$ and $g\mu$ do not renormalize.

Finally we can switch off the gauge parameter a by letting it go to zero. In this case, the analysis can be repeated obtaining the same results for the non-renormalization of the

ghost fields. However, instead of integrating the anti-ghost equations (A.11) over the five-dimensional space, we can also integrate only over the four-dimensional Lorentz invariant manifold. This implies that the anti-ghost equations (A.12), rewritten with integrals over four dimensions, are

$$\begin{aligned}
\int d^4x \frac{\delta\Gamma}{\delta\lambda(x)} &= \int d^4x \left\{ \partial_5 A'_5 + \left[\bar{\lambda}, \frac{\delta\Gamma}{\delta l} \right] + \left[\bar{\mu}, \frac{\delta\Gamma}{\delta \bar{c}} \right] + [\mu', \mu] + [\lambda', \lambda] \right. \\
&\quad \left. + \sum_{\phi} [\phi', \phi] + \left[\phi^*, \frac{\delta\Gamma}{\delta \phi''} \right] \right\}, \\
\int d^4x \frac{\delta\Gamma}{\delta c(x)} &= \int d^4x \left\{ \partial_5 A''_5 + \left[\bar{c}, \frac{\delta\Gamma}{\delta l} \right] + \left[\bar{\mu}, \frac{\delta\Gamma}{\delta \bar{\lambda}} \right] + \left[\lambda, \frac{\delta\Gamma}{\delta \mu} \right] + [c', \lambda] \right. \\
&\quad \left. + \sum_{\phi} [\phi'', \phi] + \left[\phi^*, \frac{\delta\Gamma}{\delta \phi'} \right] \right\}, \\
\int d^4x \frac{\delta\Gamma}{\delta \mu(x)} &= \int d^4x \left\{ -\partial_5 A_5^* + \left[\bar{\mu}, \frac{\delta\Gamma}{\delta l} \right] - [\lambda^*, \lambda] - c' + \sum_{\phi} (-)^{N_s+N_w} [\phi, \phi^*] \right\}.
\end{aligned} \tag{A.15}$$

They are local in the fifth component of the 5-dimensional space and satisfy all the commutation relations among themselves and with the other functional operators with the proper obvious modifications. However, due to the presence of new terms like $\partial_5 \int d^4x A'_5$, not killed by the integration over the 5-dimensional space, the commutation relation of the first equation of (A.15) with \mathcal{W}_Γ , or the second one with \mathcal{S}_Γ , generates the new functional equation

$$\begin{aligned}
& -\partial_5 \int d^4x \frac{\delta\Gamma}{\delta A_5} + \int d^4x \left\{ \left[\bar{\lambda}, \frac{\delta\Gamma}{\delta \lambda} \right] + \left[\bar{\mu}, \frac{\delta\Gamma}{\delta \bar{\mu}} \right] + \left[\bar{c}, \frac{\delta\Gamma}{\delta \bar{c}} \right] + \left[\lambda, \frac{\delta\Gamma}{\delta \mu} \right] + \left[\mu, \frac{\delta\Gamma}{\delta \mu} \right] \right. \\
& \quad \left. + \sum_{\phi} \left[\phi, \frac{\delta\Gamma}{\delta \phi} \right] + \left[\phi', \frac{\delta\Gamma}{\delta \phi'} \right] + \left[\phi'', \frac{\delta\Gamma}{\delta \phi''} \right] + \left[\phi^*, \frac{\delta\Gamma}{\delta \phi^*} \right] + [c', \mu] \right\} = 0.
\end{aligned} \tag{A.16}$$

This equation implements the invariance under gauge transformations, local in the fifth component, and, automatically, implies that the fifth component of the gauge field A_5 is not renormalized, or equivalently, by rescaling properly the gauge fields by the gauge coupling g , the combination gA_5 is not renormalized. We may use the gA_5 propagator to define an invariant charge in QCD. A similar situation happens for the abelian sector

of the Standard Model quantized in the background gauge [26].

To conclude the present section, we would like to stress that for a generic a the content of the symmetry is so rich that it constrains the form of the action strongly. Indeed, by a tedious algebra, one can easily prove that the only solution of the system which satisfies the power-counting constraints is given by the action (2.27).

Appendix B. Landau Gauge Limit

We wish to consider the important limiting case in which the gauge parameter $a \rightarrow 0$, and the 4-vector potential A_μ becomes transverse $\partial_\mu A_\mu = 0$. Like the Coulomb gauge condition $\partial_i A_i = 0$ in the 4-dimensional formulation, this gauge condition is a singular in the present 5-dimensional formulation because it leaves t -dependent but x -independent gauge-transformations $g(t)$ unfixed (cf. eq. (A.16)). However it is expected that the limit $a \rightarrow 0$ is finite in the sense that the renormalized correlation functions calculated at positive a have a finite limit as $a \rightarrow 0$. Indeed this property has been established in the 4-dimensional formalism when the Coulomb gauge is approached from an interpolating gauge [23]. Without attempting here to establish the finiteness of the limit $a \rightarrow 0$ in the present case, we shall instead derive some of its properties.

The transversality condition $\partial_\mu A_\mu = 0$ is generally called the Landau gauge. However this is not a well-defined gauge because of the Gribov ambiguity, and there is really an infinite class of Landau gauges. We shall show that in the limit $a \rightarrow 0$ we end up in a gauge, which we call the “minimal” Landau gauge, in which the additional condition

$$M(A) \equiv -\partial_\mu D_\mu(A) \geq 0 , \quad (\text{B.1})$$

is satisfied. This states that the Faddeev–Popov operator $M(A)$ is non-negative. It is a condition on A_μ that defines the (first) Gribov region. By contrast, the Faddeev–Popov method in 4 dimensions does not restrict the functional integral in the Landau gauge to the Gribov region. Indeed, if one attributes a non-perturbative significance to the Faddeev–Popov formula by BRST quantization, then a signed sum over the entire region both inside and outside the Gribov horizon is implied.

To study the limit $a \rightarrow 0$, we rescale $t \rightarrow at$ in (3.3) and obtain

$$I = \int d^5x [(2a)^{-1}(\partial_5 A_\mu - D_\mu \partial \cdot A)^2 + (a/2)(D_\lambda F_{\lambda\mu})^2] , \quad (\text{B.2})$$

where $\partial \cdot A \equiv \partial_\mu A_\mu$. As a approaches 0, the first term dominates and the probability gets concentrated at its absolute minimum namely at

$$\partial_5 A_\mu = D_\mu \partial \cdot A . \quad (\text{B.3})$$

With $t = x_5$ as the time variable, this equation defines a flow that is tangent to the gauge orbit, with infinitesimal generator $\omega = \partial \cdot A$. The *global*, non-perturbative character of this flow will be deduced from the fact that it describes steepest descent of the functional

$$F = (1/2) \int d^4x A_\mu^2 \quad (\text{B.4})$$

restricted to the gauge orbit. For under an arbitrary infinitesimal gauge transformation $\delta A_\mu = D_\mu \omega$, the functional undergoes the infinitesimal variation

$$\delta F = \int d^4x A_\mu \delta A_\mu = \int d^4x A_\mu D_\mu \omega = - \int d^4x \partial_\mu A_\mu \omega, \quad (\text{B.5})$$

so steepest descent of this functional restricted to the gauge orbit is indeed achieved with $\omega = \partial_\mu A_\mu$. Since the functional is bounded below, the steepest descent necessarily approaches a local minimum.³ At a minimum, the functional is stationary, which yields the Landau gauge condition $\partial \cdot A = 0$. Moreover its second variation is non-negative at a minimum, which, from (B.5), gives the additional condition,

$$\int d^4x \omega (-\partial \cdot D)\omega \geq 0 \text{ for all } \omega, \quad (\text{B.6})$$

which establishes (B.1). Thus in the limit $a \rightarrow 0$, the gauge-fixed functional integral (B.2), which is local in 5 dimensions, has the 4-dimensional property of concentrating the probability within the Gribov horizon.⁴

Remarkably, the Landau gauge limit just obtained from the *local* 5-dimensional action coincides with the “minimal” Landau gauge that is used in numerical studies of lattice gauge theory. Indeed in these studies, the gauge is fixed by numerically minimizing a lattice analog of the functional $F = (1/2) \int d^4x A_\mu^2$ restricted to the gauge orbit, so that the configuration is likewise brought to a local minimum of the functional, namely to a point within the first Gribov region. Thus the 5-dimensional formulation provides a BRST-invariant, renormalizable, continuum description of the minimal Landau gauge that is accessible numerically. Moreover other numerically accessible gauges such as the minimal Coulomb gauge or maximal abelian gauge may be introduced in the 5-dimensional formalism by imposing the 5-dimensional gauge condition $aA_5 = -G(x)F$, where F is an appropriately chosen functional, and $G(x) = -D_\mu \delta F / \delta A_\mu$ is the gradient in the gauge orbit direction.

³ This argument is rigorous in the 5-dimensional compact lattice-gauge formulation [1]. The classical descent could in principle end at a saddle-point but quantum fluctuations prevent this outcome.

⁴ Because there are in general more than one relative minimum on a gauge orbit, the restriction to the Gribov region (B.1) is not a complete gauge fixing. However the validity of the present formulation does not in any way depend on how the probability may be distributed among the various possible relative minima in the limit $a \rightarrow 0$, nor, for that matter, does it depend on taking the limit $a \rightarrow 0$ at all, but rather is valid for any value of $a \geq 0$.

These are not in the class of Faddeev–Popov gauges in which at the non-perturbative level a signed sum over the entire region outside the Gribov region is implied. Needless to say, if one wishes to compare numerically gauge-fixed quantities such as propagators with analytic predictions such as, for example, the Nielsen identities [27], they should be derived in the same gauge.

Appendix C. Confinement Scenario

The present 5-dimensional formulation provides a simple confinement scenario in the Landau gauge limit. The basic idea is that gA_5 , which is a Lorentz scalar, provides a vehicle for the transmission of long-range correlations that correspond to a confining force. In this respect it resembles the component gA_4 of the gluon field in the Coulomb gauge. In fact the scenario was originally developed for the Coulomb gauge by Gribov [18] and elaborated more recently [19]. However in the present formulation, the mechanism respects manifest 4-dimensional Lorentz invariance. The discussion that follows is non-perturbative, and could be based on the lattice-gauge formulation of the 5-dimensional formalism presented in [1].

It is shown in Appendix A that in the Landau gauge limit, the field gA_5 is invariant under renormalization

$$gA_5 = g_r A_{5r}, \quad \text{for } a = 0. \quad (\text{C.1})$$

This implies that in the Landau gauge limit all correlation functions of gA_5 are renormalization-group invariants. As such they are finite and independent of the ultra-violet cut-off, and depend only on the QCD scale Λ_{QCD} . Thus the statement that they are long range refers to the QCD scale.

To derive the properties of the Landau-gauge limit, we return to the first-order action (3.2), and integrate out the ghosts λ and $\bar{\lambda}$, which again gives a constant determinant. We pose $a = 0$, and obtain

$$I = \int d^5x [ib_\mu F_{5\mu} + (1/2)b_\mu^2 + (1/2)(D_\lambda F_{\lambda\mu})^2 - il\partial_\mu A_\mu] . \quad (\text{C.2})$$

Apart from the substitution $B_i^2 \rightarrow (D_\lambda F_{\lambda\mu})^2$, this is the Coulomb-gauge action in 5 dimensional space-time, and we shall solve the constraints just as is in the Coulomb gauge. We integrate out the Lagrange multiplier field l , which gives $\delta(\partial_\mu A_\mu)$, and imposes the

gauge constraint $A_\mu = A_\mu^{\text{tr}}$. Integration on A_5 which appears in the action only in the term $b_\mu F_{5\mu} = b_\mu(\partial_5 A_\mu - D_\mu A_5)$ gives $\delta(D_\mu b_\mu)$ which imposes a form of Gauss's law,

$$D_\mu b_\mu = 0 \quad (\text{C.3})$$

on the 4-dimensional color-electric field b_μ . Indeed if we write this as

$$\partial_\mu b_\mu = \rho, \quad (\text{C.4})$$

the color-charge density $\rho = j_5 = -[A_\mu, b_\mu]$ is the j_5 -component of the conserved Noether current (j_μ, j_5) of global gauge invariance. To solve Gauss's law, we write $b_\mu = b_\mu^{\text{tr}} - \partial_\mu \phi$, where ϕ plays the role of a Coulomb potential for the 4-dimensional color-electric field b_μ . Gauss's law is solved by

$$\phi = M^{-1} \rho_l, \quad (\text{C.5})$$

where $\rho_l \equiv -[A_\mu^{\text{tr}}, b_\mu^{\text{tr}}]$ is the color-charge density of the transverse gluons. Here $M = M(A^{\text{tr}}) = -D_\mu(A^{\text{tr}})\partial_\mu$, with $\mu = 1, \dots, 4$, is the 4-dimensional Faddeev–Popov operator. Its inverse $M^{-1}(A^{\text{tr}})$ is an integral operator that acts *instantaneously* in the 5-dimensional space-time and transmits the long-range force that is expected to be confining.

When this expression for ϕ is substituted back into the action, it assumes the canonical form⁵

$$I(A^{\text{tr}}, b^{\text{tr}}) = \int dt [i b_\mu^{\text{tr}} \partial_5 A_\mu^{\text{tr}} + H(A^{\text{tr}}, b^{\text{tr}})], \quad (\text{C.6})$$

with “hamiltonian”

$$\begin{aligned} H = & \int d^4x [(1/2)(b_\mu^{\text{tr}})^2 + (1/2)(D_\lambda F_{\lambda\mu})^2] \\ & + (1/2) \int d^4x d^4y \rho_l(x) [M^{-1}(-\partial^2)M^{-1}](x, y) \rho_l(y). \end{aligned} \quad (\text{C.7})$$

This is the Coulomb hamiltonian in one extra space dimension. The last term represents the instantaneous color-Coulomb interaction of separated color charge ρ_l .

⁵ One might expect that the integral over ϕ will produce the inverse of the Faddeev–Popov determinant $\det^{-1}(-\partial_\mu D_\mu)$ which, with $\mu = 1, \dots, 4$, is ill-defined in a 5-dimensional functional integral. However the integral over ϕ should be performed before setting gauge parameter $a = 0$, which gives instead $\det^{-1}(aD_5 - \partial_\mu D_\mu)$. This is a constant for the same reason that the ghost determinant is a constant, namely the propagator of the *parabolic* operator $a\partial_5 - \partial_\mu^2$ is retarded, so all closed loops vanish.

The preceding derivation may also be used to show that the propagator

$$D_{55}(x, t) = \langle gA_5(x, t)gA_5(0, 0) \rangle . \quad (\text{C.8})$$

develops an instantaneous part in the Landau gauge limit,

$$D_{55}(x, t) = V(x)\delta(t) + (\text{non - instantaneous}). \quad (\text{C.9})$$

Here $V(x)$ is an analog of the instantaneous color Coulomb potential in one extra dimension that is given by

$$V(x - y) = \langle [M^{-1}(-\partial^2)M^{-1}](x, y) \rangle . \quad (\text{C.10})$$

As shown in Appendix A, the correlation functions of gA_5 are renormalization-group invariants including in particular the instantaneous part $V(x)$, and its fourier transform $\tilde{V}(k)$. It is thus of the form $\tilde{V}(k) = g^2(k/\Lambda_{QCD})/k^2$, where $g(k/\Lambda_{QCD})$ is a running coupling constant that depends only the QCD mass scale.

Thus the Landau gauge limit of the 5-dimensional formulation exhibits all the features of the Coulomb gauge in one extra space dimension, with the additional advantage of maintaining manifest 4-dimensional Lorentz invariance. Consequently the arguments for confinement in the Coulomb gauge [18], [19], may be taken over wholesale to the present case. Without reproducing these arguments here we recall that the restriction to the first Gribov region $M(A^{\text{tr}}) \geq 0$, demonstrated in the previous section, produces 2 related effects. (i) Configurations corresponding to small eigenvalues of $M(A^{\text{tr}})$ are favored by entropy. This makes the Green function $[M^{-1}(A^{\text{tr}})](x - y)$ long range, and consequently also the instantaneous color-Coulomb potential in (C.7) and (C.10). (ii) The low frequency components of A_μ^{tr} are suppressed, thereby eliminating the physical gluons from the physical spectrum. Indeed it has been proven in lattice gauge theory in the Landau or Coulomb gauge that at infinite lattice volume the gluon propagator $D_{\mu\nu}(k)$ of transverse gluons vanishes for any probability distribution that is restricted to the first Gribov region [28]. It would require detailed dynamical arguments to determine whether the instantaneous color-Coulomb potential is sufficiently long range to confine all colored objects. However the mechanism of a long-range force that couples universally to color charge (C.7), is clearly present.

Appendix D. Spinor action and propagator

The five-dimensional formalism can accommodate the coupling of the gauge field to spinors, and moreover we observed heuristically in [1] that this extended point of view naturally introduces many of the ingredients that look rather ad hoc in the genuine four-dimensional formulation when one tries to answer non-perturbative questions. We wish to address these issues in more detail, particularly the question of spinors and chirality.

The action that defines the propagators and the interactions of the four-dimensional spinors in the five-dimensional formalism was introduced in [1]. Our postulate is that one must extend the definition of the topological BRST s -operator to spinors (in a way that does not necessitate the introduction of a Langevin equation with a fermionic noise.) Naturally this must be done in a way which is consistent with the symmetries such as the gauge invariance implemented by w -symmetry, the chirality, the conservation of the fermionic number, etc. Once this is done, the principle of locality and renormalizability must determine the form of a renormalizable local action in five dimensions. As we will see shortly, this simple five-dimensional point of view turns out to be surprisingly predictive.

Let $q(x)$ be a spinor in four dimensions. For concreteness we suppose it describes a quark, with spinor and color indices suppressed as usual. Extend it to a five-dimensional object $q(x) \rightarrow q(x, t)$, without affecting the spinorial index. In accordance with our postulate, q is a member of a spinor quartet, made up of $q, \Psi_q, \bar{\Psi}_q$, and b_q , which are, besides the anti-commuting spinor q , its commuting topological ghost and anti-ghost, and b_q , the anti-commuting Lagrange multiplier. They each carry unit quark charge, and s and w act on them according to

$$\begin{aligned} (s + w)q + (c + \lambda)q &= \Psi_q & (s + w)\Psi_q + (c + \lambda)\Psi_q &= \Phi q \\ (s + w)\bar{\Psi}_q + (c + \lambda)\bar{\Psi}_q &= b_q & (s + w)b_q + (c + \lambda)b_q &= \Phi \bar{\Psi}_q, \end{aligned} \tag{D.1}$$

where c, λ and Φ act on the spinors in (say) the fundamental representation, $c \equiv c^a t^a$, where t^a are the Gell-Mann matrices. By separate conservation of N_s and N_w , this gives $sq = \Psi_q - cq$, $s\Psi_q = -c\Psi_q - \Phi q$ and $s\bar{\Psi}_q = -c\bar{\Psi}_q + b_q$, $sb_q = -c, b_q - \Phi, \bar{\Psi}_q$, and the w -symmetry acts on the field as an infinitesimal gauge transformation with a parameter equal to the ghost λ . There is a corresponding independent quartet for the anti-quark which consists of $q^\dagger, \Psi_q^\dagger, \bar{\Psi}_q^\dagger$, and b_q^\dagger , each with quark charge -1 .⁶

⁶ Since we already use “bar” notation to designate an anti-ghost, such as $\bar{\Psi}$, we unconventionally use “dagger” notation, such as q^\dagger instead of \bar{q} , to designate Dirac conjugation.

Five-dimensional Lorentz covariance of spinors $q, \Psi_q, \bar{\Psi}_q, b_q$ does not concern us since we only have in view an $SO(4) \times R$ -invariant theory rather than an $SO(5)$ -invariant one, namely the 5-dimensional formulation of the standard Dirac action,

$$S = \int d^4x \, q^\dagger (\gamma_\mu D_\mu - m) q. \quad (D.2)$$

The method of stochastic quantization is not of direct relevance for the case of anticommuting fields. Therefore, to determine the 5-dimensional action for spinors, we postulate that it must be invariant under the topological BRST symmetry (D.1). Using power counting and locality requirements, this must give:

$$I_q = \int d^5x \, s \{ \bar{\Psi}_\mu \frac{\delta S}{\delta A_\mu} + \bar{\Psi}_q^\dagger [D_5 q - K(\frac{\delta S}{\delta q^\dagger} - \frac{1}{2} b_q)] + [-q^\dagger D_5 - (\frac{\delta S}{\delta q} + \frac{1}{2} b_q^\dagger) K] \bar{\Psi}_q \} , \quad (D.3)$$

that is,

$$I_q = \int d^5x \, s \{ \bar{\Psi}_\mu q^\dagger \gamma_\mu q + \bar{\Psi}_q^\dagger \{ [D_5 - K(\gamma_\mu D_\mu - m)] q + \frac{K}{2} b_q \} + \{ q^\dagger [-D_5 - (\gamma_\mu D_\mu - m) K] - b_q^\dagger \frac{K}{2} \} \bar{\Psi}_q \}. \quad (D.4)$$

The first term contains the anti-ghost $\bar{\Psi}_\mu$ of A_μ , and is an additional contribution to the gluon action (2.27).

Here is K is a kernel that in principle is at our disposal, because the expectation value of observables calculated on a given time-slice at equilibrium is independent of K as can be seen by Ward identities. However we must choose it to obtain a well-defined functional integral. In particular one easily sees that it is necessary that the combination $K(-\gamma_\mu D_\mu + m)$ be bounded below. If we give the standard canonical dimension 3/2 to q and its ghost Ψ_q , and canonical dimension 5/2 to the anti-ghost $\bar{\Psi}_q$ and Lagrange multiplier b_q , and similarly for the q^\dagger quartet, and impose gauge-invariance in the form of w -invariance, then power counting, $SO(4)$ -invariance and gauge invariance imply that

$$K = \gamma^\mu D_\mu + M. \quad (D.5)$$

This is the most general action which one may write down with these properties, to within renormalization constants, apart from the fact that the same kernel K multiplies both

$\gamma_\mu D_\mu$ and b_q . As a matter of convenience we shall set the mass m in the kernel equal to the mass of the Dirac spinor, $M = m$ so the combination

$$Q \equiv K(-\gamma_\mu D_\mu + m) = KK^\dagger = K^\dagger K = (-\gamma_\mu D_\mu + m)K = -(\gamma_\mu D_\mu)^2 + m^2 \quad (\text{D.6})$$

is hermitian and positive and chirally even. (For well-defined perturbation theory, $mM \geq 0$ is sufficient.)

We may consider Weyl spinors in four dimensions, with chiral coupling to gauge fields. The signal that we deal with Weyl spinors is minimally contained in the range of values over which the spinor index runs, together with the various $V \pm A$ coupling that may arise in the interactions. The four-dimensional matrix γ^5 extends trivially to the fifth γ matrix in five dimensions.

The extension to chiral gauge coupling is automatic in the 5-dimensional fermion action (D.3), if the original 4-dimensional Dirac action (D.2) is chiral. In particular, one sees that if q is, say, left-handed, then Ψ_q is also left-hand, whereas $\bar{\Psi}_q$ and b_q are right-handed, and oppositely for the anti-quark quartet.

Let us find the form of the free propagators. Because of the presence of the kernel there is a mixing of q and $ba - q$. The quadratic approximation of the action is:

$$I_q = \int d^4x dt [b_q^\dagger (\partial_5 - \partial_\mu \partial_\mu + m^2) q - q^\dagger (-\partial_5 - \partial_\mu \partial_\mu + m^2) b_q + b_q^\dagger (\gamma_\mu \partial_\mu + m) b_q + \text{ghost terms}] \quad (\text{D.7})$$

This gives the following matrix propagator, between the independent pairs (q^\dagger, b_q^\dagger) and (q, b_q) :

$$\begin{pmatrix} \frac{m + i\gamma_\mu p_\mu}{\omega^2 + (p^2 + m^2)^2} & \frac{1}{i\omega + p^2 + m^2} \\ \frac{1}{-i\omega + p^2 + m^2} & 0 \end{pmatrix} \quad (\text{D.8})$$

If we integrate the $q^\dagger - q$ matrix element over ω , we obtain the usual free Dirac propagator

$$\frac{1}{\pi} \int d\omega \frac{m + i\gamma_\mu p_\mu}{\omega^2 + (p^2 + m^2)^2} = \frac{m + i\gamma_\mu p_\mu}{p^2 + m^2}. \quad (\text{D.9})$$

There are closed loops of fields q and q^\dagger , but the associated ghosts Ψ_q and $\bar{\Psi}_q$ have retarded propagators so are no closed ghost loops, just as in the bose case. The topological quantum field theory in five dimensions operates for the spinors and their topological ghosts and Lagrange multipliers in the same way as it does for gauge fields. We have a well-defined perturbation theory for the system of spinors and gauge fields in five dimensions, and the familiar arguments about renormalizability apply.

Appendix E. Anomalies

The five-dimensional theory is a local quantum quantum field theory that is renormalizable by power counting. Our aim is that the generating functional of Green functions must be constructed from the requirement of s and w invariances. Then, observables are defined from the operators that are the cohomology with ghost number zero of w , with expectation values taken at equal times.

The possibility of renormalizing the theory in five dimensions, while maintaining the two symmetries can be jeopardized by the presence of anomalies. Such anomalies can destroy the theory in two different ways, since the s -symmetry ensures the existence of a Fokker–Plank equation and the w -symmetry ensures that the drift forces give gauge invariant observables. Our knowledge of the four-dimensional theory indicates that some kind of breakdown must occur in the five-dimensional approach when one couples a chiral fermion to the theory, or a set of chiral fermions without couplings that ensure anomaly compensations.

A primary check is thus to verify whether radiative corrections can break the Ward identities of s and w symmetries. This can be done by computing the possible obstructions of the Ward identity in five dimensions and the values of their coefficient. But, as already mentioned, the currents and their conservation laws in four and five dimensions are not related in an obvious way. For instance, if one calls $J_\mu(x)$ a conserved current of the four-dimensional theory, the insertion of $\partial_\mu J_\mu(x, t)$ in correlation functions with fields and/or operators taken at different values of the fifth time t has no reason to give zero.

Since we cannot rely on the above approach to understanding the effect of the anomaly, we are led to investigate the possibility that, when one computes insertions of four-dimensionally conserved currents in Green functions, their conservation is not fulfilled in the limit of equal fifth times for all arguments of the Green function.

The search of anomalies relies therefore on an indirect argument. Knowing from general principles that the symmetry of the theory in a given slice is the ordinary four-dimensional gauge symmetry, we can select and compute in five dimensions the possible anomalous vertices the four-dimensional theory. We will do so, and find an ambiguity in the equal all fifth time limit for the external legs. We will see that this ambiguity is rooted in the usual problem of computing Feynman diagrams with superficial linear divergences.

Before doing this important calculation, we will nevertheless show the existence of a cocycle for the combined w and s symmetries. Its overall coefficient vanishes perturbatively (which indicates that no anomaly occurs in five dimensions, for all Green functions).

However, this cocycle has an interesting form, and its role in quantum field theory could be important. We could not find its interpretation, but it is striking that it is related in a formal way to the usual four-dimensional consistent anomaly by descent equations.

The possible obstructions to the Ward identity for the w and s symmetries in five dimensions must be 5-forms, with ghost number (1,0) and (0,1) respectively. We call them $\Delta_5^{(1,0)}$ and $\Delta_5^{(0,1)}$. The ghost unification allows us to define $\Delta_5^1 = \Delta_5^{(1,0)} + \Delta_5^{(0,1)}$. Since $(s + w)^2 = 0$, Δ_5^1 must satisfy the consistency equation for the s and w symmetries:

$$(s + w)\Delta_5^1 + d\Delta_4^2 = 0 \quad (\text{E.1})$$

If we assume that only exterior products play a role, we can easily manipulate this equation. We use that $(s + w)d\Delta_5^1 = 0$, and thus $d\Delta_5^1 + (s + w)\Delta_6^0 = 0$. Thus $(s + w)d\Delta_6^0 = 0$. Since $d\Delta_6^0$ has ghost number 0, it is a 7-form that can only depend on A and dA , and since the symmetry is $(s + w)A = \Psi - D(c + \lambda)$, the only possibility is that $d\Delta_6^0 = 0$, which implies in turn that Δ_6^0 is an invariant polynomial in the Yang–Mills curvature $F = dA + AA$, that is, $\Delta_6^0 = \text{Tr}(FFF) = dQ_5(A, F)$, where $Q_5(A, F)$ is a Chern class of rank 5. We thus have that the solution of the Wess and Zumino consistency equation (E.1) is obtained from the piece with ghost number 2 in the following identity satisfied by $\Delta_6(F + \Psi + \Phi) = \text{Tr}[(F + \Psi + \Phi)^3]$:

$$(s + w + d)\Delta_6(F + \Psi + \Phi) = 0 \quad (\text{E.2})$$

We thus have:

$$\begin{aligned} w\text{Tr}(\Psi FF) &= 0 \\ (s + w)\text{Tr}(\Psi FF) + d\text{Tr}(\Psi \Psi F + FF\Phi) &= 0 \end{aligned} \quad (\text{E.3})$$

and thus

$$\begin{aligned} \Delta_5^{1,0} &= \text{Tr}(\Psi FF) \\ \Delta_5^{0,1} &= 0 \end{aligned} \quad (\text{E.4})$$

is a candidate for the anomaly.

On the other hand, due to the Chern–Simons formula,

$$\text{Tr}[(F + \Psi + \Phi)^3] = (d + s + w)Q_5(A + c + \lambda, F + \Psi + \Phi) \quad (\text{E.5})$$

we have

$$\text{Tr}(\Psi FF) = d\text{Tr}\left[c\frac{\delta}{\delta A}|_F Q_5(A, F) + \Psi\frac{\delta}{\delta F}|_A Q_5(A, F) + sQ_5(A, F)\right] \quad (\text{E.6})$$

Therefore, in five dimensions, $\text{Tr}(\Psi FF)$ is locally the sum of d - and s - exact terms. However, although it is w -invariant, it is not w -exact, up to d - and s -exact terms, and therefore we can identify $\text{Tr}(\Psi FF)$ as part (and probably the unique element) of the cohomology with ghost number one for the w -symmetry. Due to this fact, we can probably safely call $\text{Tr}(\Psi FF)$ the consistent five dimensional anomaly.

One recognizes among all terms in the right hand side of (E.6) the interesting piece: $\partial_t \Delta_4^1$, where

$$\text{Tr} \Delta_4^1 = \text{Tr} \left[c \frac{\delta}{\delta A} \Big|_F Q_5(A, F) \right] \quad (\text{E.7})$$

is nothing else than the consistent four-dimensional anomaly that can be directly related to the ABBJ triangle anomaly once it is inserted on the right hand side of the Ward identity in four dimensions. All this suggests that adding $Q_5(A, F)$ could become an interesting issue.

Although, it is certainly natural to interpret the existence of the cocycle $\text{Tr}(\Psi FF)$ as the origin of the anomaly that must occur when the theory is coupled to chiral four-dimensional spinors, we have not been able yet to see the way $\text{Tr}(\Psi FF)$ plays a role in the five-dimensional theory. This does not mean however that the cocycle doesn't play a role, for instance in Fujikawa type manipulations when one reduces the theory in four dimensions.

Consider now practical computations, to understand how the anomaly will manifest itself. We can take the case of a single spinor q , and introduce the vector and axial currents $J^\mu = \bar{q} \gamma^\mu q$ and $J_5^\mu = \bar{q} \gamma^5 \gamma^\mu q$. The anomaly question amounts to compute the form factor of the 1PI vertex in four dimensions $T^{\alpha\beta} = \langle K_\mu J_5^\mu(K), J^\alpha(k), J^\beta(k') \rangle$ that is proportional to $\epsilon_{\alpha\beta\gamma\delta} k^\gamma k'^\delta$. Here we assume that the vector current is conserved, and $K = k + k'$.

The free propagators of spinors in five dimensions are given by (D.8). $T^{\alpha\beta}(k, k', T, t, t')$ is thus given by the following 5-dimensional Feynman integral

$$\begin{aligned} & \int d\Omega d\omega d\omega' d^4p \exp i(\Omega T + \omega t + \omega' t') \\ & \text{tr} \left(K_\mu \gamma^\mu \gamma^5 \frac{(p-k)_\rho \gamma^\rho}{\omega^2 + ((p-k)_\rho (p-k)^\rho)^2} \right. \\ & \left. \gamma^\alpha \frac{p_\rho \gamma^\rho}{\Omega^2 + (p_\rho p^\rho)^2} \gamma^\beta \frac{(p+k')_\rho \gamma^\rho}{\omega'^2 + ((p+k')_\rho (p+k')^\rho)^2} \right) \end{aligned} \quad (\text{E.8})$$

We can use the translation invariance along the fifth time direction, and set T at the origin of time. It is thus sufficient to compute the integral at $T = 0$. We can thus investigate

carefully the various way one can approach the value $T = t = t' = 0$ that must give the result of the four-dimensional theory.

We first observe, that if we set brutally $T = t = t' = 0$ the integral (E.8) is just equal to the ordinary four-dimensional triangle diagram. This is easily seen by performing the integration over $d\Omega$, $d\omega$ and $d\omega'$ by using Cauchy theorem for picking out the poles in Ω , ω and ω' , which gives:

$$T^{\alpha\beta}(k, k', T = t = t') = \int d\Omega d\omega d\omega' d^4p \operatorname{tr} \left(K_\mu \gamma^\mu \gamma^5 \frac{(p-k)_\rho \gamma^\rho}{\omega^2 + ((p-k)_\rho (p-k)^\rho)^2} \gamma^\alpha \right. \\ \left. \frac{p_\rho \gamma^\rho}{\Omega^2 + (p_\rho p^\rho)^2} \gamma^\beta \frac{(p+k')_\rho \gamma^\rho}{\omega'^2 + ((p+k')_\rho (p+k')^\rho)^2} \right) \quad (\text{E.9})$$

When one extracts the term proportional to $\epsilon_{\alpha\beta\mu\nu} k^\mu k'^\nu$, this expression has the well known linear divergence which provides a non-vanishing value for the anomaly.

If, on the other hand, we set t and t' different from zero, the divergence in the integration over p is regularized due to the exponentials in t . We can compute the integral explicitly for small values of t and t' , and one finds that the anomaly coefficient is proportional to:

$$\epsilon_{\alpha\beta\mu\nu} k^\mu k'^\nu \frac{|t-T| + |t'-T|}{|t-T| + |t'-T| + |t-t'|} . \quad (\text{E.10})$$

This expression exhibits an ambiguity in the equal-time limit $t = t' = T$. The ratio of absolute values varies between 1/2 and 1. It equals unity if one first sets $t = t'$ and then $t = T$; but it equals 1/2 if one first sets $t = T$ and then $t' = T$. This contrasts with the tree-level property that the correlation functions in four dimensions are obtained as a smooth limit of their counterparts in five dimensions.

The ambiguity of the expression (E.10) is related to the property that first setting $t = t' = T$ causes superficially linear divergences. We thus foresee that only the triangle can lead to an ambiguity in the limit $t = t' = T$; otherwise the fifth time acts an invariant regulator.

Appendix F. Absence of fermion doubling

We have seen in our analysis of anomalies that they do not appear in the 5-dimensional theory per se, but rather when the fifth time is restricted to a slice. This suggests that the absence of fermion doubling in the 5-dimensional formulation is practically automatic, and does not require the introduction of domain walls [20], [21], [22].

Indeed, consider the matrix form of the free Dirac action (D.7) between the independent pairs (q^\dagger, b_q^\dagger) and (q, b_q) ,

$$\begin{pmatrix} 0 & \partial_5 + \partial^2 - m^2 \\ \partial_5 - \partial^2 + m^2 & \gamma_\mu \partial_\mu + m \end{pmatrix}, \quad (\text{F.1})$$

where $\partial^2 \equiv \partial_\mu \partial_\mu$ for $\mu = 1, \dots, 4$, is the 4-dimensional lattice Laplacian. We shall use a standard lattice discretization of the operators that appear here, $\partial_5 \rightarrow (\partial_5)_d$ etc., where the discretized operators are defined by

$$\begin{aligned} (\partial_5)_d q(x, t) &\equiv \frac{1}{2} [q(x, t+1) - q(x, t-1)] \\ (\partial^2)_d q(x, t) &\equiv \sum_{\mu=1}^4 [q(x + \hat{\mu}, t) + q(x - \hat{\mu}, t) - 2q(x, t)] \\ (\partial_\mu)_d q(x, t) &\equiv \frac{1}{2} [q(x + \hat{\mu}, t) - q(x - \hat{\mu}, t)], \end{aligned} \quad (\text{F.2})$$

where $\hat{\mu}$ is a unit vector in the $+\mu$ -direction. We have preserved hermiticity properties, so $(\partial_5)_d$ and $(\partial_\mu)_d$ are anti-symmetric whereas $(\partial^2)_d$ is symmetric. The trick is that we discretized the 4-dimensional lattice laplacian ∂^2 and the lattice derivatives ∂_μ independently, so $(\partial^2)_d \neq \sum_{\mu=1}^4 [(\partial_\mu)_d]^2$. It will turn out that we never have to invert the Dirac operator but only the even chirality operators that appear in the off-diagonal matrix elements of the matrix (F.1). This simplifying property results from our choice of the kernel, $K = m + \gamma_\mu D_\mu$, corresponding to $M = m$.

All these operators are diagonalized by lattice Fourier transformation, and the matrix (F.1) becomes in terms of lattice momenta θ_μ and θ_5 ,

$$\begin{pmatrix} 0 & i \sin \theta_5 - Q_0 \\ i \sin \theta_5 + Q_0 & K_0 \end{pmatrix}, \quad (\text{F.3})$$

where $Q_0 \equiv m^2 + \sum_{\mu=1}^4 2(1 - \cos \theta_\mu)$, and $K_0 \equiv i \sin \theta_\mu \gamma_\mu + m$.

The propagator is given by the inverse matrix,

$$\begin{pmatrix} \frac{K_0}{\sin^2 \theta_5 + Q_0^2} & \frac{1}{i \sin \theta_5 + Q_0} \\ \frac{1}{i \sin \theta_5 - Q_0} & 0 \end{pmatrix}. \quad (\text{F.4})$$

The free 4-dimensional lattice $q^\dagger - q$ propagator in momentum space is obtained at equal fifth time, namely

$$S(\theta_\mu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta_5 \frac{K_0}{\sin^2 \theta_5 + Q_0^2} = \frac{K_0}{Q_0(1 + Q_0^2)^{1/2}}. \quad (\text{F.5})$$

In the continuum limit, this integral gets contributions from the neighborhood of $\theta_5 = 0$ and $\theta_5 = \pi$ which reflects fermion doubling in θ_5 . However there is no doubling of the physical 4-dimensional propagator. Indeed in the continuum limit, $Q_0^2 = O(a^4)$ is negligible compared to 1, where a is the lattice spacing, and we obtain

$$S(\theta_\mu) \approx \frac{K_0}{Q_0} = \frac{m + i \sum_{\mu=1}^4 \sin \theta_\mu \gamma_\mu}{m^2 + \sum_{\mu=1}^4 2(1 - \cos \theta_\mu)}. \quad (\text{F.6})$$

No fermion doubling occurs here. As asserted, the Dirac operator is never inverted, but only the operator $i \sin \theta_5 + m^2 + \sum_{\mu=1}^4 2(1 - \cos \theta_\mu)$ that has even chirality. The denominator of the last expression cannot be factorized on the lattice, $Q_0 \neq K_0 K_0^\dagger$, but it does factorize in the continuum limit, when $\sin \theta_\mu \rightarrow q_\mu$, and $2(1 - \cos \theta_\mu) \rightarrow q_\mu^2$.

As regards practical numerical simulation, a possible advantage of the lattice discretization described here with respect to domain-wall fermions is that every hyperplane $x_5 = \text{const}$ may be used for 4-dimensional fermions, not just the one domain wall [20], [21], [22].

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